The shallow water approximation applied to the aluminium electrolysis process

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Abstract

The industrial production of aluminium is an electrolysis process where two superposed horizontal liquid layers are subjected to a mainly vertical electric current supplied by carbon electrodes. The lower layer consists of molten aluminium and lies on the cathode. The upper layer is the electrolyte and is covered by the anode. The interface between the two layers is often perturbed, leading to oscillations, or waves, similar to the waves on the surface of seas or lakes. The presence of electric currents and the resulting magnetic field are responsible for electromagnetic (Lorentz) forces within the fluid, which can amplify these oscillations and have an adverse influence on the process.

The electrolytic bath vertical to horizontal aspect ratio is such, that it is advantageous to use the shallow water equations to model the interface motion. These are the depth-averaged Navier–Stokes equations, commonly used in the modelling of oceans and rivers. Different orders of approximations may be adopted in averaging the Navier–Stokes equations so that nonlinear and dispersion terms may be taken into account. Although these terms are essential to the prediction of wave dynamics, they are neglected in most of the literature on interface instabilities in aluminium reduction cells where only the linear theory is usually considered. The unknown variables are the two horizontal components of the fluid velocity, the height of the interface and the electric potential.

In this application, a finite volume resolution of the double-layer shallow water equations including the electromagnetic sources has been developed, for incorporation into a generic three-dimensional computational fluid dynamics code that also deals with heat transfer within the cell.

1 Background in Shallow Water Theory

1.1 Surface Waves

The depth-averaged Navier–Stokes equations are known as the shallow water equations. These equations give an approximation for the dynamics of long waves of length l and small amplitude a at the surface a shallow layer of fluid of depth h_0. They express the conservation of mass and momentum and can be written in dimensional form in terms of the gravity g, the depth-averaged velocity $\mathbf{u}$ and displacement η of the surface height $h = h_0 + \eta$. Their dimensionless form is expressed in terms of the small parameters $\alpha = a/h_0$ and $\beta = h_0^2/l^2$;

$$\eta_t + \nabla \cdot [(1 + \alpha\eta)\mathbf{u}] = 0, \tag{1}$$

$$\mathbf{u}_t + \alpha\mathbf{u} \cdot \nabla\mathbf{u} + \nabla\eta - \frac{1}{3}\beta\nabla\nabla \cdot \mathbf{u}_t = O(\alpha^2, \alpha\beta), \tag{2}$$

the subscript t denoting partial time derivatives. The velocity $\mathbf{u}$ is scaled with ga/c_0, $c_0 = \sqrt{gh_0}$ being the linear phase velocity, the time t with l/c_0, the horizontal coordinate with l and the vertical displacement η with a. Neglecting the small terms in α and β gives a linear theory of waves which has been used by almost all authors of interface instabilities in aluminium reduction cells, e.g. Sneyd [1]. The theories of nonlinear waves is reviewed in several books including Whitham [2] and Mei [3].

1.2 Two-layers Theory

Tomasson & Melville [4] have shown that the waves at the interface between two layers of fluids of similar densities can be represented by the same Boussinesq equations. Using the notations chosen by these authors, the top and bottom layers are respectively numbered 1 and 2 as shown on figure 1. Their densities are ρ_1, ρ_2 and their depths h_1, h_2. After introducing the equivalent single layer depth $h_0 = h_1\, h_2/(h_1 + h_2)$ and the square of the linear phase speed $c_0^2 = gh_0\Delta\rho/\rho_0$ where $\Delta\rho = \rho_2 - \rho_1$ and $\rho_0 = (\rho_1 + \rho_2)/2$ is a reference density, and assuming a balance between the volume fluxes $\mathbf{U}_1 = (h_1 - \eta)\mathbf{u}_1$ and $\mathbf{U}_2 = (h_2 + \eta)\mathbf{u}_2$ in the two layers, the single flux $\mathbf{U}$ defined as $\mathbf{U} = \mathbf{U}_2 = -\mathbf{U}_1$ satisfies the Boussinesq equations for the displacement η of the interface between the two fluids. Indeed, in dimensionless form and in terms of the small parameters

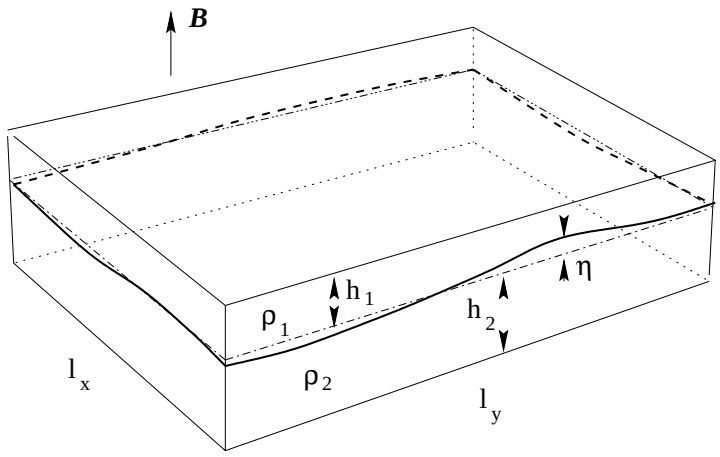

Figure 1: The two-layers of fluid in an electrolytic bath and the perturbation $\eta(x, y)$ of the interface.

$\alpha = a/h_0$ and $\beta = h_0^2/l^2$, scaling h_1 and h_2 with h_0, the densities ρ_1 and ρ_2 with ρ_0, the velocity flux $\mathbf{U}$ with $h_2 g_0 a/c_0$, the time t with l/c_0, the coordinates x and y with l and the vertical displacement η with a,

$$\eta_t + \nabla \cdot \mathbf{U} = 0, \tag{3}$$

$$\mathbf{U}_t + \alpha \left(\frac{1}{h_2} - \frac{1}{h_1} \right) [\mathbf{U} \cdot \nabla \mathbf{U} - (\eta \mathbf{U})_t]$$

$$+ \nabla \eta - \frac{1}{3} \beta h_1 h_2 \nabla \nabla \cdot \mathbf{U}_t = O(\alpha^2, \alpha\beta). \tag{4}$$

2 Application to an Electrolytic Bath

An aluminium reduction cell is submitted to a strong electric current density $\mathbf{J}$ crossing both layers of fluid from the upper carbon anode to the lower carbon cathode. A significant magnetic field $\mathbf{B}$ mainly created by the input and output currents provided through "bus bars" to these two electrodes is also present. Any change in the position of the interface between the two liquid gives rise to a perturbation $\mathbf{j}$ in the current distribution as the electrical path in the electrolyte of poor conductivity σ_1 is either decreased or increased as shown on figure 2. Let us define $\Delta\Phi$ as the electric potential drop across the electrolyte.

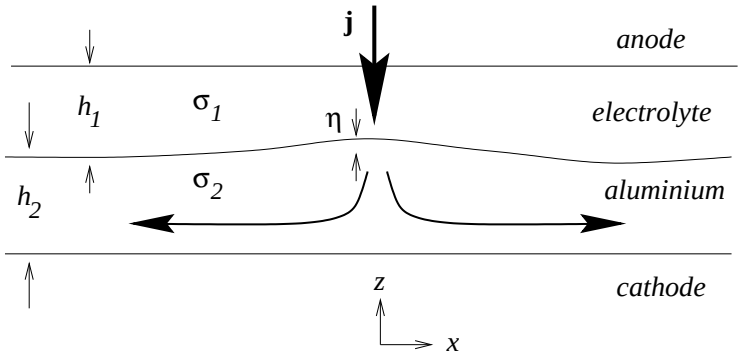

Figure 2: Perturbation of the electric currents due to the deformation of the interface between the molten aluminium and the electrolyte. The current takes the shortest path through the poorly conducting electrolyte.

The vertical current density perpendicular to the interface is then defined in this fluid as

$$\mathbf{J}_\perp = -\sigma_1 \nabla_\perp \Phi = \sigma_1 \frac{\Delta\Phi}{h_1 - \eta} \sim \sigma_1 \frac{\Delta\Phi}{h_1}\left(1 + \frac{\eta}{h_1}\right) = \mathbf{J}_{0_\perp} + \mathbf{j}_\perp \qquad (5)$$

The contribution of the interface perturbation in the normal current density $\mathbf{J}_\perp$ is therefore

$$\mathbf{j}_\perp = \frac{\sigma_1 \Delta\Phi}{h_1^2}\eta \qquad (6)$$

while the constant vertical current without deformation of the interface is just $\mathbf{J}_{0_\perp} = \sigma_1 \Delta\Phi / h_1$. The conservation law for electric charges may be written as

$$\nabla \cdot \mathbf{J} = \nabla \cdot \mathbf{j} = \nabla_\perp \cdot \mathbf{j}_\perp + \nabla_{||} \cdot \mathbf{j}_{||} = 0, \qquad (7)$$

and may be integrated over the depth of the molten aluminium layer as

$$\int_{cathode}^{interface} \nabla \cdot \mathbf{j}\, \mathrm{d}z = \nabla \cdot \int_{cathode}^{interface} \mathbf{j}_{||}\, \mathrm{d}z + \int_{cathode}^{interface} \frac{\partial \mathbf{j}_\perp}{\partial z}\, \mathrm{d}z =$$

$$\nabla_{||} \cdot \langle \mathbf{j}_{||} \rangle \times \int_{cathode}^{interface} \mathrm{d}z + [\mathbf{j}_\perp]_{cathode}^{interface} = (h_2 + \eta)\, \nabla_{||} \cdot \langle \mathbf{j}_{||} \rangle + \mathbf{j}_\perp^{interface} = 0, \quad (8)$$

where $\langle \mathbf{j}_{||} \rangle = -\sigma_2 \nabla_{||} \Phi$ is the depth-averaged current density in the molten aluminium and can be solved from the Poisson equation

$$\nabla_{||}^2 \Phi(x, y) = -\frac{\sigma_1}{\sigma_2} \frac{\Delta\Phi}{h_1^2 h_2} \eta(x, y). \qquad (9)$$

Since the electromagnetic force gives some energy and momentum to the fluid, the modelled interface oscillations can only increase if the dissipation due to the viscosity of the fluid is not taken into account. The linear friction law used by Bojarevics [5] is introduced in the model.

2.1 Scaling

Let us introduce the small parameter $\delta = (\rho_2 - \rho_1)/(\rho_2 + \rho_1) = \Delta\rho/(2\rho_0)$ and the aspect ratio $\gamma = l_y^2/l_x^2$ of the bath horizontal dimensions so that the x- and y-coordinates are now scaled with l_x and l_y respectively. The time is scaled with l_x/c_0, the x- and y-components U and V of the velocity flux thus being scaled with $\alpha c_0 h_0$ and $\alpha\sqrt{\gamma}c_0 h_0$. The linear friction law in $-\kappa\mathbf{U}$ used by Bojarevics [5] with a friction coefficient κ proportional to $\alpha\beta^{-1}\mathrm{Re}^{-1}$ is used. Defining the interaction parameter $N = (l_x^3\sigma_1\Delta\Phi B_{0\perp})/(l_y h_1^2\rho_2 c_0^2)$ characterizing the ratio of electromagnetic to inertial forces and scaling the electric potential Φ with $(\sigma_1 l_x^2 a\Delta\Phi)/(\sigma_2 h_1^2 h_2)$ the dimensionless Boussinesq equations are

$$\eta_t + U_x + \gamma V_y = 0, \tag{10}$$

$$\left(1+\delta\frac{h_1-h_2}{h_1+h_2}\right)U_t + \alpha\left(\frac{1}{h_2}-\frac{1}{h_1}\right)\left[(UU)_x + \gamma(VU)_y - \eta U_t\right]$$
$$+\eta_x - \frac{1}{3}\beta h_1 h_2\left(U_{xxt} + \gamma V_{xyt}\right) = -N\Phi_y - \kappa U + O(\alpha^2, \alpha\beta, \alpha\delta, \beta\delta), \tag{11}$$

$$\gamma\left(1+\delta\frac{h_1-h_2}{h_1+h_2}\right)V_t + \alpha\gamma\left(\frac{1}{h_2}-\frac{1}{h_1}\right)\left[(UV)_x + \gamma(VV)_y - \eta V_t\right]$$
$$+\gamma\eta_y - \frac{1}{3}\beta\gamma h_1 h_2\left(U_{xyt} + \gamma V_{yyt}\right) = N\Phi_x - \gamma\kappa V + O(\alpha^2, \alpha\beta, \alpha\delta, \beta\delta), \tag{12}$$

the subscripts x and y denoting partial spatial derivatives.

3 Numerical method and Simulations

The finite volume method with a fully staggered grid as shown on figure 3 is used. All terms are computed with second order central finite differences for both spatial and time derivatives. The mass conservation equation (10) and the momentum equation (12) are solved

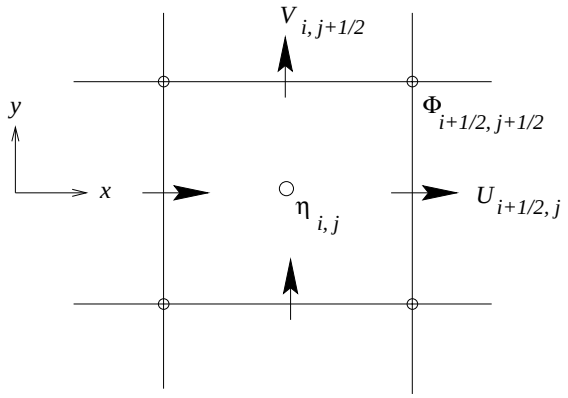

Figure 3: Finite volume mesh.

alternatively. The perturbation η of the interface is computed explicitly at the time step n from its value at the time step $n-1$ and from the momentum (U, V) at the intermediate time step $n-1/2$:

$$\eta_{i,j}^{n} = \eta_{i,j}^{n-1} + \Delta t \left(\frac{U_{i+1/2,j}^{n-1/2} - U_{i-1/2,j}^{n-1/2}}{\Delta x} + \gamma \frac{V_{i,j+1/2}^{n-1/2} - V_{i,j-1/2}^{n-1/2}}{\Delta y} \right), \quad (13)$$

Then the electric potential can be solved from eq. (9) using a classical five-point finite difference scheme for the Laplacian of Φ and a four-point averaged value of $\eta_{i+1/2,j+1/2}$. The momentum U is computed implicitly at the time step $n+1/2$: The momentum U is computed implicitly at the time step $n+1/2$:

$$\begin{aligned} U_{i+1/2,j}^{n+1/2} = U_{i+1/2,j}^{n-1/2} + \Delta t \left[1 + \left(\frac{1}{h_2} - \frac{1}{h_1} \right) \left(\delta - \alpha \eta_{i+1/2,j}^{n} \right) \right]^{-1} \\ \left\{ -\alpha \left(\frac{1}{h_2} - \frac{1}{h_1} \right) \left[(UU)_x |_{i+1/2,j}^{n} + \gamma \, (VU)_y |_{i+1/2,j}^{n} \right] \right. \\ - \frac{\eta_{i+1,j}^{n} - \eta_{i,j}^{n}}{\Delta x} + \frac{1}{3} \beta h_1 h_2 \left(U_{xxt} |_{i+1/2,j}^{n} + \gamma \, V_{xyt} |_{i+1/2,j}^{n} \right) \\ \left. - N B_{\perp i+1/2,j}^{n} \frac{\Phi_{i+1/2,j+1/2}^{n} - \Phi_{i+1/2,j-1/2}^{n}}{\Delta y} - \kappa U_{i+1/2,j}^{n} \right\} \end{aligned} \quad (14)$$

where the x-component $(UU)_x + (VU)_y$ of the advection term is discretised in conservative form using a control volume centred at

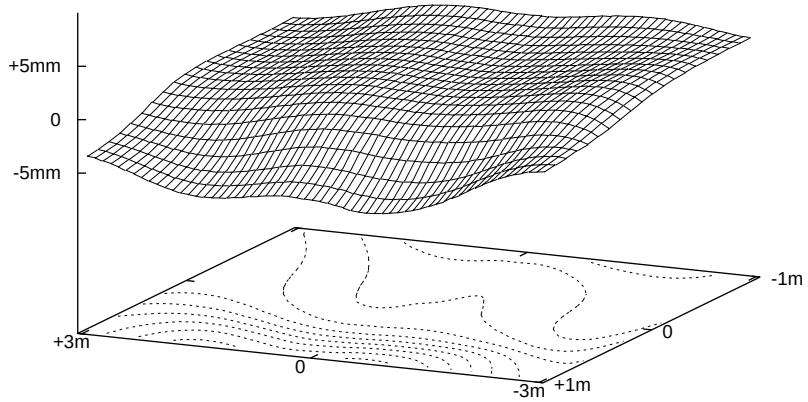

Figure 4: Example of the position $\eta(x, y)$ of the interface between the molten aluminium and the electrolyte after amplification of a small sine wave by the electromagnetic forces within the fluid.

$i + 1/2, j$ and is averaged over the time steps $n - 1/2$ and $n + 1/2$, while the dispersion terms U_{xxt} and V_{xyt} are computed with second order central finite differences over two and three points for first and second order derivatives. The code is run with the parameters of the aluminium production plant in Slatina, Romania. The bath is 6×2 meters. The electrolyte and aluminium layers are 5 and 25 cm respectively. The imposed magnetic field is given by a commercial finite elements software [6]. The maximum value of the vertical component of the magnetic field is 10^{-2} T and the electric potential drop in the electrolyte is 1.5 V.

4 Conclusion

The shallow layer equations discretised in finite volumes are successfully applied to model interface perturbations in an aluminium elec-

trolysis cell. This finite volume formulation allows the integration of this model with standard computational fluid dynamics software for heat transfer computations, a work which is in progress now.

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